

Scale Factor Dualities in Anisotropic Cosmologies

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Abstract

The concept of scale factor duality is considered within the context of the spatially homogeneous, vacuum Brans–Dicke cosmologies. In the Bianchi class A, it is found that duality symmetries exist for the types I, II, VI₀, VII₀, but not for types VIII and IX. The Kantowski–Sachs and locally rotationally symmetric Bianchi type III models also exhibit a scale factor duality, but no such symmetries are found for the Bianchi type V. In this way anisotropy and spatial curvature may have important effects on the nature of such dualities.

PACS NUMBERS: 04.50.+h, 11.30.Ly, 95.30.Sf, 98.80.Hw

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1 Introduction

The concept of *scale factor duality* has played a central role in string cosmology in recent years [1, 2, 3, 4]. If one assumes a spatially isotropic and flat Friedmann–Robertson–Walker (FRW) universe, the genus–zero effective superstring action is invariant under the discrete Z_2 transformation [1]

$$\bar{a} = a^{-1}, \quad \bar{\Phi} = \Phi - 6 \ln a \quad (1.1)$$

that inverts the cosmological scale factor, a , and shifts the value of the dilaton field, Φ . Eq. (1.1) maps an expanding cosmological solution onto a contracting one, and vice-versa. However, since the theory is also invariant under time reversal, $\bar{t} = -t$, the contracting solution may be mapped onto a new, expanding solution. Furthermore, the Hubble expansion parameter $H \equiv d \ln a / dt$ is invariant under the simultaneous application of scale factor duality and time reversal, but its first derivative changes sign. This implies that the new solution will represent a superinflationary cosmology characterized by the conditions $\dot{a} > 0$, $\ddot{a} > 0$ and $\dot{H} > 0$ if the original solution describes a decelerated expansion with $\dot{a} > 0$ and $\ddot{a} < 0$, and vice-versa, where a dot denotes differentiation with respect to cosmic time.

It is this feature that forms the basis of the *pre–big bang scenario* [1, 5]. In this picture, the universe was initially in a state of low curvature and weak coupling. The non–minimal coupling of the dilaton field to the graviton, $g_{\mu\nu}$, drives an epoch of superinflation that represents the pre–big bang phase of the universe’s history. The universe evolves into a regime of high curvature and strong–coupling and to lowest–order in the string effective action, the end state is a singularity in both the curvature and the coupling [6]. However, it is anticipated that such a singular state will be avoided when higher–order and quantum effects are included [7, 8, 9]. The conjecture is that this will result in a smooth transition to the (duality related) post–big bang, decelerating solution.

Since one of the main advantages of inflationary models in general is that they can

explain the high degree of homogeneity, isotropy and spatial flatness in the observable universe, it is of importance to study the pre-big bang scenario in more general settings other than that of the spatially flat FRW cosmology. In the standard, chaotic inflationary scenario, such generalizations may have qualitatively significant consequences [10]. Recently, Turner and Weinberg have studied the role of spatial curvature within the context of the FRW universes [11]. They have argued that for certain initial conditions, spatial curvature may reduce the duration of the superinflationary expansion to such an extent that the horizon problem cannot be solved. Veneziano and collaborators, on the other hand, have considered the general effects of anisotropy and inhomogeneity on the scenario under the assumptions of small initial curvature and weak coupling and have shown that superinflationary expansion is possible in most regions of the universe [12, 13].

The purpose of this paper is to investigate whether the idea of scale factor duality may be extended to the spatially homogeneous and anisotropic Bianchi cosmologies [14, 15, 16, 17]. This is interesting firstly because small anisotropies will inevitably be present in the real universe and secondly in view of the central role that scale factor duality plays in the pre-big bang scenario. Bianchi cosmologies represent the simplest deviations from the FRW environment and they therefore allow us to gain insight into what may happen as we deviate from the idealized FRW settings.

Within the context of string theory, the scale factor duality, Eq. (1.1), is embedded within the symmetry group $O(3,3)$ of the anisotropic and spatially flat Bianchi type I background [2, 4]. More generally, there exists a target space duality in the σ -model whenever there is an abelian symmetry [18]. The group of duality transformations is $O(d,d)$ when d abelian isometries are present. (For a recent review see, e.g., Ref. [3]). The procedure for establishing target space duality was generalized to encompass non-abelian isometries in Ref. [19] and considered further in Refs. [20, 21, 22, 23]. Gasperini, Ricci and Veneziano discussed the spatially homogeneous cosmologies of arbitrary dimension and, in

particular, the dual to a Bianchi type V cosmological solution [20]. However, they found that the dual model did not satisfy the requirements for conformal invariance (vanishing beta-functions) even when arbitrary shifts in the dilaton field were allowed. Such a problem arises because the isometry group is non-semisimple and this results in a trace anomaly that cannot be absorbed by the dilaton [23, 24]. The non-local nature of the anomalous term in the Bianchi V solution was investigated by Elitzur *et al.* [21]. More general questions regarding the inverse transformation [25] and the quantum equivalence of the dual theories have recently been addressed [26] and non-abelian duals of the Taub-NUT space have been derived in Refs. [24, 27].

The emphasis of the present paper is different in that we focus on the existence of dualities in the Brans-Dicke theory of gravity [28]. When the antisymmetric two-form potential vanishes, the one-loop order beta-functions of string theory may be viewed as the field equations derived from the dilaton-graviton sector of the Brans-Dicke action, where the coupling constant between the dilaton and graviton is given by $\omega = -1$ [29]. We consider all possible values of ω , thereby placing the results from string cosmology into a wider context. Since we only consider the existence of scale factor dualities in this theory, we do not include the possible effects of the two-form potential. We also derive the conditions necessary for scale factor duality under the assumption that the dual and original spacetimes have the same isometry group. This differs from previous analyses in string theory, where in general the isometries of the dual background are different from those of the original.

The Brans-Dicke theory is interesting for other reasons. It is the simplest example of a scalar-tensor theory [30] and represents a natural extension of Einstein gravity. The compactification of $(4 + d)$ -dimensional Einstein gravity on an isotropic, d -dimensional torus results in a Brans-Dicke action, where the dilaton is related to the radius of the internal space and $\omega = -1 + 1/d$ [31]. Finally, the Brans-Dicke theory is consistent with the observational limits from primordial nucleosynthesis [32] and weak-field solar system

experiments [33] when $\omega > 500$ and it therefore represents a viable theory of gravity in this region of parameter space.

The spatially flat, D -dimensional FRW Brans–Dicke cosmology exhibits a scale factor duality for all $\omega \neq -D/(D-1)$ that reduces to Eq. (1.1) when $\omega = -1$ and $D = 4$ [34]. Cadoni has recently studied the dualities of $(2+1)$ -dimensional Brans–Dicke gravity and found that there exists a continuous $O(2)$ symmetry in the theory when the metric corresponds to the three-dimensional equivalent of the Bianchi type I universe [35]. In this paper we find that the Bianchi type I Brans–Dicke cosmology possesses a continuous $O(3)$ symmetry and show that this is restricted to an $O(2)$ symmetry for the type II model and to a discrete duality for types VI_0 and VII_0 . The Kantowski–Sachs and locally rotationally symmetric (LRS) Bianchi type III models also exhibit a scale factor duality. On the other hand, no such symmetries exist for types V, VIII, and IX.

The paper is organised as follows. In section 2 we briefly summarize the important features of the spatially homogeneous spacetimes and derive the reduced Brans–Dicke actions for these models. In section 3, we derive the symmetries of the types I, II, VI_0 and VII_0 universes. Types V, VIII and IX are investigated in section 4 together with the Taub universe, the Kantowski–Sachs model and the LRS Bianchi type III cosmology. We conclude in section 5.

2 Spatially Homogeneous Brans–Dicke Cosmologies

The gravitational sector of the Brans–Dicke theory of gravity is [28]

$$S = \int d^4x \sqrt{-g} e^{-\Phi} \left[R - \omega (\nabla\Phi)^2 - 2\Lambda \right], \quad (2.1)$$

where R is the Ricci curvature scalar of the spacetime with metric $g_{\mu\nu}$ and signature $(-, +, +, +)$, $g \equiv \det g_{\mu\nu}$, Φ is the dilaton field and Λ is the cosmological constant.

Bianchi models are a class of spatially homogeneous cosmologies whose metrics admit

a three-dimensional Lie group of isometries G_3 that acts simply-transitively on three-dimensional space-like hypersurfaces of homogeneity [14, 15, 16, 17]. The line element for these cosmologies may be written as

$$ds^2 = -dt^2 + h_{ab}\omega^a\omega^b, \quad a, b = 1, 2, 3, \quad (2.2)$$

where t represents cosmic time, $h_{ab} = h_{ab}(t)$ is the metric on the surfaces of homogeneity and ω^a are the corresponding one-forms for each Bianchi type. If a three-form $\epsilon_{abc} = \epsilon_{[abc]}$ is specified on the Lie algebra of G_3 , the antisymmetric structure constants $C^a_{bc} = -C^a_{cb}$ may be written as $C^a_{bc} = m^{ad}\epsilon_{dbc} + \delta^a_{[b}a_{c]}$, where $m^{ab} = m^{ba}$, $a_c \equiv C^a_{ac}$ and indices are raised and lowered with h^{ab} and h_{ab} , respectively. The Jacobi identity $C^a_{b[c}C^b_{de]} = 0$ then implies that $m^{ab}a_b = 0$, i.e., that a_b must be transverse to m^{ab} . The Lie algebra is in the Bianchi class A if $a_b = 0$ and the class B otherwise [15]. A basis may be chosen without loss of generality where m^{ab} has a diagonal form with components ± 1 or 0 and $a_b = (A, 0, 0)$.

For the Bianchi class A, the scalar curvature, $^{(3)}R$, of the three-surfaces is uniquely determined by the structure constants of the Lie algebra of G_3 [36]:

$$\begin{aligned} ^{(3)}R &= -C^a_{ab}C^c{}_c{}^b + \frac{1}{2}C^a_{bc}C^c{}_a{}^b - \frac{1}{4}C_{abc}C^{abc} \\ &= -h^{-1} \left(m_{ab}m^{ab} - \frac{1}{2}m^2 \right), \end{aligned} \quad (2.3)$$

where $h^{-1} \equiv \det h^{ab}$. In the case of diagonal Bianchi models, the three-metric may be parametrized in the form

$$h_{ab}(t) = e^{2\alpha(t)} \left(e^{2\beta(t)} \right)_{ab}, \quad (2.4)$$

where

$$\beta_{ab} = \text{diag} \left[-2\beta_+, \beta_+ - \sqrt{3}\beta_-, \beta_+ + \sqrt{3}\beta_- \right] \quad (2.5)$$

is a traceless matrix that quantifies the anisotropy (shape change) of the models. The ‘averaged’ scale factor $a(t) \equiv e^{\alpha(t)} = [\det h_{ab}]^{1/6}$ determines the behaviour of the effective

spatial volume of the universe. The functions $a_i(t)$ defined by $h_{ab} \equiv \text{diag}(a_1^2, a_2^2, a_3^2)$ have the form

$$\begin{aligned} a_1 &= \exp[\alpha - 2\beta_+] \\ a_2 &= \exp[\alpha + \beta_+ - \sqrt{3}\beta_-] \\ a_3 &= \exp[\alpha + \beta_+ + \sqrt{3}\beta_-] \end{aligned} \quad (2.6)$$

and may be considered as the cosmological scale factors in the different directions of anisotropy. The FRW cosmologies represent the isotropic limits of the Bianchi types I (and VII₀) for $k = 0$, type V for $k = -1$ and type IX for $k = +1$.

The second-order field equations for the Bianchi class A cosmologies may be consistently derived from a point Lagrangian by substituting the *ansatz* (2.2) into Eq. (2.1) and integrating over the spatial variables. This class includes types I, II, VI₀, VII₀, VIII and IX. Assuming that the dilaton is constant on the surfaces of homogeneity, we find that

$$S = \int dt e^{3\alpha - \Phi} \left[-6\dot{\alpha}^2 + 6\dot{\alpha}\dot{\Phi} + \omega\dot{\Phi}^2 + 6\dot{\beta}_+^2 + 6\dot{\beta}_-^2 + {}^{(3)}R(\alpha, \beta_{\pm}) - 2\Lambda \right], \quad (2.7)$$

where a boundary term has been ignored. The action for each Bianchi type A cosmology is therefore uniquely determined by the functional form of the three-curvature (2.3).

The Lagrangian formulation of the field equations for the class B models is ambiguous because there exist spatial divergence terms whose variations do not vanish, with the consequence that the variational principle does not always result in the correct field equations [17, 37, 38]. As a result, we will not discuss the class B models in detail. However, one class B model where a consistent Lagrangian can be derived is the Bianchi type V with a metric of the form

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left[dx^2 + e^{2x} \left(e^{-2\sqrt{3}\beta_-(t)} dy^2 + e^{2\sqrt{3}\beta_-(t)} dz^2 \right) \right]. \quad (2.8)$$

The Brans–Dicke action is formally given by Eq. (2.7) with $\beta_+ = 0$, where the three–

curvature is

$${}^{(3)}R = -6e^{-2\alpha}. \quad (2.9)$$

There also exists the Kantowski–Sachs model corresponding to the unique spatially homogeneous universe with no simply-transitive G_3 group of isometries [39]. The G_3 acts multiply-transitively on two-dimensional surfaces of maximal symmetry and positive curvature. The metric of the Kantowski–Sachs spacetime is

$$ds^2 = -dt^2 + a_1^2(t)dr^2 + a_2^2(t)d\Omega_2^2, \quad (2.10)$$

where $d\Omega_2^2$ is the metric on the unit two-sphere. The LRS Bianchi type III universe with $m^a_a = 0$ may be viewed as the analytic continuation of the Kantowski–Sachs metric, where $d\Omega_2^2$ is replaced by the metric on a compact two-dimensional manifold of constant negative Ricci curvature -2 . If we express the two scale factors as

$$\begin{aligned} a_1 &\equiv e^{\alpha-2\beta} \\ a_2 &\equiv e^{\alpha+\beta}, \end{aligned} \quad (2.11)$$

the action (2.1) becomes

$$S = \int dt e^{3\alpha-\Phi} \left[-6\dot{\alpha}^2 + 6\dot{\alpha}\dot{\Phi} + \omega\dot{\Phi}^2 + 6\dot{\beta}^2 + {}^{(3)}R(\alpha, \beta) - 2\Lambda \right], \quad (2.12)$$

where the curvature potential is given by

$${}^{(3)}R = 2ke^{-2\alpha-2\beta} \quad (2.13)$$

and $k = +1, -1, 0$ for the Kantowski–Sachs, LRS Bianchi type III and axisymmetric Bianchi type I models, respectively.

3 Symmetries of the Bianchi Class A Action

Throughout this paper we consider linear transformations on the variables appearing in Eqs. (2.7) and (2.12). The symmetries of the action (2.7) become manifest by rewriting it

in terms of a *shifted scale factor*

$$\chi \equiv \sqrt{\frac{3}{4+3\omega}} [\alpha + (1+\omega)\Phi] , \quad (3.1)$$

a *shifted dilaton field*

$$\sigma \equiv \kappa^{-1} (\Phi - 3\alpha) , \quad \kappa \equiv \sqrt{\frac{4+3\omega}{3+2\omega}} \quad (3.2)$$

and the rescaled anisotropy parameters

$$\beta_1 \equiv \sqrt{6}\beta_+, \quad \beta_2 \equiv \sqrt{6}\beta_- . \quad (3.3)$$

Throughout we shall take $\omega > -3/2$ and exclude the particular value of $\omega = -4/3$. The averaged scale factor and dilaton are given by

$$\alpha = -\frac{(1+\omega)}{\sqrt{(3+2\omega)(4+3\omega)}}\sigma + \frac{1}{\sqrt{3(4+3\omega)}}\chi \quad (3.4)$$

$$\Phi = \frac{1}{\sqrt{(3+2\omega)(4+3\omega)}}\sigma + \sqrt{\frac{3}{4+3\omega}}\chi \quad (3.5)$$

and action (2.7) then takes the form

$$S = \int dt e^{-\kappa\sigma} \left[-\dot{\sigma}^2 + \dot{\chi}^2 + \dot{\beta}_1^2 + \dot{\beta}_2^2 + {}^{(3)}R(\chi, \sigma, \beta_i) - 2\Lambda \right] . \quad (3.6)$$

We remark that the theory is invariant under time reversal, $\bar{t} = -t$.

The action for the FRW cosmologies is given by Eq.(3.6), where $\beta_i = 0$ and ${}^{(3)}R = 6ke^{-2\alpha}$. The kinetic sector of the action is then invariant under a discrete Z_2 scale factor duality [1, 34]

$$\bar{\chi} = -\chi, \quad \bar{\sigma} = \sigma. \quad (3.7)$$

Eq. (3.4) implies that the three-curvature is not invariant under (3.7) and the duality is therefore broken in the spatially curved isotropic universes. We now proceed to investigate the symmetries of the Bianchi class A cosmologies.

3.1 Bianchi Type I

When the cosmological constant is non-zero, any symmetry of the theory that leads to non-trivial transformations of the dynamical degrees of freedom must leave the shifted dilaton field invariant, i.e., $\bar{\sigma} = \sigma$. This implies that the symmetries involve the remaining three variables, $\{\chi, \beta_i\}$.

The eigenvalues of m^{ab} are all zero for the Bianchi type I and the three-curvature therefore vanishes. It follows from Eq. (3.6) that the action is symmetric under a continuous O(3) transformation. The standard realization for the action of this group on the variables $\{\chi, \beta_i\}$ is given by

$$\bar{V} = RV, \quad V \equiv \begin{pmatrix} \beta_2 \\ \beta_1 \\ \chi \end{pmatrix}, \quad (3.8)$$

where

$$R \equiv \pm \begin{pmatrix} -s(\gamma)s(\psi) + c(\gamma)c(\psi)c(\theta) & -c(\gamma)c(\theta)s(\psi) - s(\gamma)c(\psi) & c(\gamma)s(\theta) \\ -c(\gamma)s(\psi) + c(\psi)s(\gamma)c(\theta) & -s(\gamma)c(\theta)s(\psi) - c(\gamma)c(\psi) & s(\gamma)s(\theta) \\ -s(\theta)c(\psi) & s(\theta)s(\psi) & c(\theta) \end{pmatrix}, \quad (3.9)$$

$\{\theta, \gamma, \psi\}$ are the three Euler angles, $s(x) \equiv \sin x$ and $c(x) \equiv \cos x$. The group O(3) has two connected components comprising the subgroup SO(3), which is isomorphic to the group of proper rotations in three dimensions, and the *set* of reflections. It is isomorphic to $SO(3) \times G_2$, where G_2 is the matrix group of order 2 with elements $\{1_3, -1_3\}$ and 1_3 is the 3×3 identity matrix. Hence, the proper rotations of O(3) are given by the positive sign in Eq. (3.9) and the reflections are given by the negative sign.

Substituting Eqs. (3.8) and (3.9) into Eqs. (3.3)–(3.5) implies that the averaged scale factor, dilaton and anisotropy parameters transform to

$$\begin{aligned} \bar{\alpha} = \frac{3(1+\omega) \pm \cos \theta}{4+3\omega} \alpha \pm \frac{\sqrt{8+6\omega} \sin \theta \sin \psi}{4+3\omega} \beta_+ \mp \frac{\sqrt{8+6\omega} \sin \theta \cos \psi}{4+3\omega} \beta_- \\ \pm \frac{(1+\omega)(\cos \theta \mp 1)}{4+3\omega} \Phi \end{aligned} \quad (3.10)$$

$$\bar{\Phi} = \pm \frac{3(\cos \theta \mp 1)}{4 + 3\omega} \alpha \pm \frac{3\sqrt{8 + 6\omega} \sin \theta \sin \psi}{4 + 3\omega} \beta_+ \mp \frac{3\sqrt{8 + 6\omega} \sin \theta \cos \psi}{4 + 3\omega} \beta_- \pm \frac{(3(1 + \omega) \cos \theta \pm 1)}{4 + 3\omega} \Phi \quad (3.11)$$

$$\bar{\beta}_+ = \pm \frac{\sqrt{8 + 6\omega} \sin \gamma \sin \theta}{2(4 + 3\omega)} \alpha \pm (\cos \gamma \cos \psi - \sin \gamma \cos \theta \sin \psi) \beta_+ \pm (\cos \gamma \sin \psi + \cos \psi \sin \gamma \cos \theta) \beta_- \pm \frac{\sqrt{8 + 6\omega}(1 + \omega) \sin \gamma \sin \theta}{2(4 + 3\omega)} \Phi \quad (3.12)$$

$$\bar{\beta}_- = \pm \frac{\sqrt{8 + 6\omega} \cos \gamma \sin \theta}{2(4 + 3\omega)} \alpha \mp (\cos \gamma \cos \theta \sin \psi + \sin \gamma \cos \psi) \beta_+ \pm (\cos \gamma \cos \psi \cos \theta - \sin \gamma \sin \psi) \beta_- \pm \frac{\sqrt{8 + 6\omega}(1 + \omega) \cos \gamma \sin \theta}{2(4 + 3\omega)} \Phi, \quad (3.13)$$

where the upper (lower) signs correspond to the rotations (reflections) of the $O(3)$ symmetry. The corresponding transformations for the scale factors may then be derived by substituting Eqs. (3.10)–(3.13) into Eq. (2.6).

The effect of introducing anisotropy in this fashion is to extend the discrete Z_2 duality (3.7) of the $k = 0$ FRW model to a continuous $O(3)$ symmetry. The duality (3.7) may now be interpreted as the subgroup of $O(3)$ that leaves the anisotropy parameters at zero and inverts e^χ . For example, one may specify the Euler angles to be $\theta = 2n\pi$ and $\gamma = \psi = (-1)^n(2n + 1)\pi/2$, where n is a positive integer, and consider the lower sign in Eq. (3.9), since this implies that $R = \text{diag}(1, 1, -1)$. When $R = -1_3$ and $\omega = -1$, we recover the analogue of the scale factor duality (1.1) for the Bianchi type I string cosmology, where all three scale factors are simultaneously inverted, $\bar{a}_i = a_i^{-1}$ [2, 4].

The kinetic sectors of all Brans–Dicke Bianchi class A actions are invariant under the transformations (3.10)–(3.13). However, the curvature potential (2.3) is not in general symmetric under the full group of transformations. Depending on the Bianchi type, this imposes further restrictions on the symmetries of the model.

3.2 Bianchi Type II

We now consider the Bianchi II universe, where $m^{ab} = \text{diag}(1, 0, 0)$ and the curvature potential is

$${}^{(3)}R = -\frac{1}{2} \exp[-2\alpha - 8\beta_+]. \quad (3.14)$$

Since the potential is independent of β_- , the action is symmetric under the discrete Z_2 symmetry $\bar{\beta}_- = -\beta_-$, as in the general relativistic case. This corresponds to the simultaneous interchange of the cosmological scale factors $a_2 \leftrightarrow a_3$. Applying the change of variables defined in Eqs. (3.3)–(3.5) implies that

$${}^{(3)}R(\chi, \sigma, \beta_1) = -\frac{1}{2} \exp \left[C_1 \sigma - \frac{8}{\sqrt{6}} (C_2 \chi + \beta_1) \right], \quad (3.15)$$

where

$$C_1 \equiv \frac{2(1+\omega)}{\sqrt{(3+2\omega)(4+3\omega)}} \quad (3.16)$$

$$C_2 \equiv \frac{1}{2\sqrt{8+6\omega}}. \quad (3.17)$$

The non-trivial curvature potential (3.15) imposes an extra constraint on the group of transformations that leaves the action (3.6) invariant. As well as requiring the kinetic sector to be invariant, we also require the constraint

$$C_2 \bar{\chi} + \bar{\beta}_1 = C_2 \chi + \beta_1 \quad (3.18)$$

to be satisfied.

A symmetry of the theory may be found by performing the field redefinitions

$$x \equiv C_2 \chi + \beta_1 \quad (3.19)$$

$$y \equiv \frac{2}{\sqrt{3}} \left(\frac{8+6\omega}{11+8\omega} \right)^{1/2} (\chi - C_2 \beta_1). \quad (3.20)$$

Action (3.6) transforms to

$$S = \int dt e^{-\kappa\sigma} \left[\frac{4}{3} \left(\frac{8+6\omega}{11+8\omega} \right) \dot{x}^2 + \dot{y}^2 + \dot{\beta}_2^2 - \dot{\sigma}^2 - \frac{1}{2} e^{C_1 \sigma - 8x/\sqrt{6}} - 2\Lambda \right]. \quad (3.21)$$

Thus, Eq. (3.21) is invariant under a continuous $O(2)$ transformation that acts non-trivially on the variables $\{y, \beta_2\}$ and leaves σ and x invariant. We may express the symmetry group as $O(2) = SO(2) \times B_2$, where $SO(2)$ is isomorphic to the group of rotations in two dimensions and B_2 is the cyclic group of order 2 that is represented by the two matrices 1_2 and $\text{diag}(1, -1)$, where 1_2 is the 2×2 identity matrix. A typical realization of the action of $SO(2)$ on the variables $\{y, \beta_2\}$ is given by

$$\begin{pmatrix} \bar{y} \\ \bar{\beta}_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} y \\ \beta_2 \end{pmatrix}. \quad (3.22)$$

The reflections are given by the product of the $SO(2)$ rotation matrix and the B_2 matrix. By employing Eqs. (3.19) and (3.20), together with Eqs. (3.3)–(3.5), this group of transformations may be expressed in terms of $\{\alpha, \Phi, \beta_{\pm}\}$:

$$\begin{aligned} \bar{\alpha} = & \frac{8(1+\omega)(\cos \theta - 1)}{33 + 24\omega} \Phi + \frac{(8 \cos \theta + 25 + 24\omega)}{33 + 24\omega} \alpha \\ & - \frac{4(\cos \theta - 1)}{33 + 24\omega} \beta_+ + \frac{4 \sin \theta}{\sqrt{33 + 24\omega}} \beta_- \end{aligned} \quad (3.23)$$

$$\begin{aligned} \bar{\Phi} = & \frac{(8(1+\omega) \cos \theta + 3)}{11 + 8\omega} \Phi + \frac{8(\cos \theta - 1)}{11 + 8\omega} \alpha \\ & - \frac{4(\cos \theta - 1)}{11 + 8\omega} \beta_+ + \frac{4\sqrt{3} \sin \theta}{\sqrt{11 + 8\omega}} \beta_- \end{aligned} \quad (3.24)$$

$$\begin{aligned} \bar{\beta}_+ = & \frac{2(1+\omega)(1 - \cos \theta)}{33 + 24\omega} \Phi + \frac{2(1 - \cos \theta)}{33 + 24\omega} \alpha \\ & + \frac{(32 + 24\omega + \cos \theta)}{33 + 24\omega} \beta_+ - \frac{\sin \theta}{\sqrt{33 + 24\omega}} \beta_- \end{aligned} \quad (3.25)$$

$$\begin{aligned} \bar{\beta}_- = & \mp \frac{2(1+\omega) \sin \theta}{\sqrt{33 + 24\omega}} \Phi \mp \frac{2 \sin \theta}{\sqrt{33 + 24\omega}} \alpha \\ & \pm \frac{\sin \theta}{\sqrt{33 + 24\omega}} \beta_+ \pm \cos \theta \beta_-. \end{aligned} \quad (3.26)$$

The form of Eqs. (3.23)–(3.26) implies that the symmetry only applies for $\omega > -11/8$.

The transformations (3.23)–(3.25) are the same for both components of the $O(2)$ group. The upper and lower signs in Eq. (3.26) correspond to the rotations and reflections of $O(2)$,

respectively. It follows that in the case of the $\text{SO}(2)$ component, for example, the scale factors transform to

$$\begin{aligned} \ln \bar{a}_1 = & \frac{4 \cos \theta + 7 + 8\omega}{11 + 8\omega} \alpha - \frac{2 \cos \theta + 20 + 16\omega}{11 + 8\omega} \beta_+ \\ & + \frac{2\sqrt{3} \sin \theta}{\sqrt{11 + 8\omega}} \beta_- + \frac{4(\cos \theta - 1)(1 + \omega)}{11 + 8\omega} \Phi \end{aligned} \quad (3.27)$$

$$\begin{aligned} \ln \bar{a}_2 = & \frac{2 \cos \theta + 2\sqrt{11 + 8\omega} \sin \theta + 9 + 8\omega}{11 + 8\omega} \alpha \\ & - \sqrt{3} \left(\cos \theta - \frac{\sin \theta}{\sqrt{11 + 8\omega}} \right) \beta_- - \frac{\cos \theta - 12 - 8\omega + \sqrt{11 + 8\omega} \sin \theta}{11 + 8\omega} \beta_+ \\ & + \frac{2(1 + \omega) \left(\cos \theta - 1 + \sqrt{11 + 8\omega} \sin \theta \right)}{11 + 8\omega} \Phi \end{aligned} \quad (3.28)$$

$$\begin{aligned} \ln \bar{a}_3 = & \frac{2 \cos \theta + 9 + 8\omega - 2\sqrt{11 + 8\omega} \sin \theta}{11 + 8\omega} \alpha \\ & + \sqrt{3} \left(\cos \theta + \frac{\sin \theta}{\sqrt{11 + 8\omega}} \right) \beta_- - \frac{\cos \theta - 12 - 8\omega - \sqrt{11 + 8\omega} \sin \theta}{11 + 8\omega} \beta_+ \\ & + \frac{2(1 + \omega) \left(\cos \theta - \sqrt{11 + 8\omega} \sin \theta - 1 \right)}{11 + 8\omega} \Phi. \end{aligned} \quad (3.29)$$

For the reflections, the transformation of a_1 is given by Eq. (3.27), but the transformations of a_2 and a_3 differ slightly from Eqs. (3.28) and (3.29) because they depend directly on β_- .

3.3 Bianchi Types VI_0 and VII_0

The elements of m^{ab} for the types VI_0 and VII_0 are $m^{ab} = \text{diag}(0, 1, -1)$ and $m^{ab} = \text{diag}(0, 1, 1)$, respectively. The curvature potential for each model is given by

$${}^{(3)}R = -2e^{-2\alpha+4\beta_+} f(\beta_-), \quad (3.30)$$

where

$$f(\beta_-) = \begin{cases} \cosh^2 \left(2\sqrt{3}\beta_- \right), & \text{Type } \text{VI}_0 \\ \sinh^2 \left(2\sqrt{3}\beta_- \right), & \text{Type } \text{VII}_0. \end{cases} \quad (3.31)$$

Rewriting Eq. (3.30) in terms of the shifted variables (3.1)–(3.3) implies that

$${}^{(3)}R = -2 \exp \left[C_1 \sigma - \sqrt{\frac{8}{3}} (C_3 \chi - \beta_1) \right] f(\beta_2), \quad (3.32)$$

where C_1 is given by Eq. (3.16) and

$$C_3 \equiv (8 + 6\omega)^{-1/2}. \quad (3.33)$$

A discrete symmetry of these models is uncovered by defining new variables:

$$x \equiv C_3 \chi - \beta_1 \quad (3.34)$$

$$y \equiv \frac{1}{\sqrt{1 + C_3^2}} (\chi + C_3 \beta_1). \quad (3.35)$$

Action (3.6) then takes the form

$$S = \int dt e^{-\kappa \sigma} \left[\left(\frac{8 + 6\omega}{9 + 6\omega} \right) \dot{x}^2 + \dot{y}^2 + \dot{\beta}_2^2 - \dot{\sigma}^2 - 2e^{C_1 \sigma - \sqrt{8}x/\sqrt{3}} f(\beta_2) - 2\Lambda \right] \quad (3.36)$$

and Eq. (3.36) is independent of the variable y . This implies that the Bianchi types VI₀ and VII₀ are symmetric under the discrete duality transformation

$$\bar{x} = x, \quad \bar{y} = \pm y, \quad \bar{\sigma} = \sigma, \quad \bar{\beta}_2 = \pm \beta_2. \quad (3.37)$$

Substituting these transformations into Eqs. (3.3)–(3.5) implies that

$$\bar{\alpha} = \frac{5 + 6\omega}{9 + 6\omega} \alpha - \frac{4(1 + \omega)}{9 + 6\omega} \Phi - \frac{4}{9 + 6\omega} \beta_+ \quad (3.38)$$

$$\bar{\Phi} = -\frac{4}{3 + 2\omega} \alpha - \frac{1 + 2\omega}{3 + 2\omega} \Phi - \frac{4}{3 + 2\omega} \beta_+ \quad (3.39)$$

$$\bar{\beta}_+ = -\frac{2}{9 + 6\omega} \alpha - \frac{2(1 + \omega)}{9 + 6\omega} \Phi + \frac{7 + 6\omega}{9 + 6\omega} \beta_+ \quad (3.40)$$

$$\bar{\beta}_- = \pm \beta_- \quad (3.41)$$

and the scale factors (2.6) therefore transform to

$$\bar{a}_1 = \exp [\alpha - 2\beta_+] \quad (3.42)$$

$$\bar{a}_2 = \exp \left[\frac{1 + 2\omega}{3 + 2\omega} \beta_+ + \frac{1 + 2\omega}{3 + 2\omega} \alpha - \frac{2(1 + \omega)}{3 + 2\omega} \Phi \mp \sqrt{3} \beta_- \right] \quad (3.43)$$

$$\bar{a}_3 = \exp \left[\frac{1 + 2\omega}{3 + 2\omega} \beta_+ + \frac{1 + 2\omega}{3 + 2\omega} \alpha - \frac{2(1 + \omega)}{3 + 2\omega} \Phi \pm \sqrt{3} \beta_- \right]. \quad (3.44)$$

The scale factor a_1 is invariant under the duality transformation (3.37) and a_2 and a_3 transform non-trivially. This is consistent since the curvature potential (3.30) depends directly on a_1^{-2} and we have required this term to be invariant.

It is interesting to consider the string cosmologies, where $\omega = -1$. If $\bar{\beta}_- = -\beta_-$, Eqs. (2.6), (3.43) and (3.44) imply that $\bar{a}_2 = a_2^{-1}$ and $\bar{a}_3 = a_3^{-1}$. Thus, the duality symmetry inverts these two scale factors as in the Bianchi type I model. The presence of curvature implies that the spatial volume of the universe is not directly inverted, however. The type I and VII₀ models are important because they simplify to the spatially flat FRW cosmology in the isotropic limit. We conclude that in the string case, there is a direct generalization of the scale factor duality (3.7) in the type I model, but the symmetry is more restrictive in the type VII₀.

To summarize thus far, we have found that duality symmetries exist for the Bianchi types I, II, VI₀ and VII₀. We will discuss types VIII, IX and V in the next section, together with the Kantowski–Sachs and LRS Bianchi type III models.

4 Other Bianchi Types and the Kantowski–Sachs Model

4.1 Bianchi Types V, VIII and IX

The O(3) invariance of the kinetic sector of the Bianchi class A action (3.6) implies that the linear transformations discussed in the preceding sections are specified in terms of a maximum of three parameters. Spatial curvature restricts the group of transformations that leave the full action invariant and therefore reduces the symmetry. For the type II model, one free parameter remains, corresponding to an arbitrary angle of rotation. For types VI₀ and VII₀ the curvature potential (3.30) is a sum of three independent exponential terms and requiring each of these to be invariant under a linear transformation of the variables implies that at least one degree of freedom must be lost per term. Thus, all three parameters must be uniquely specified and consequently the action can only be symmetric under a discrete

transformation.

The curvature potentials for the Bianchi types VIII and IX are

$${}^{(3)}R_{\text{VIII}} = -\frac{1}{2}e^{-2\alpha} \left[e^{-8\beta_+} + 4e^{4\beta_+} \left(\cosh 2\sqrt{3}\beta_- \right)^2 + 4e^{-2\beta_+} \sinh 2\sqrt{3}\beta_- \right] \quad (4.1)$$

$${}^{(3)}R_{\text{IX}} = -\frac{1}{2}e^{-2\alpha} \left[e^{-8\beta_+} + 4e^{4\beta_+} \left(\sinh 2\sqrt{3}\beta_- \right)^2 - 4e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- \right], \quad (4.2)$$

respectively. Both potentials contain six separate terms that represent different exponential functions of $\{\alpha, \beta_{\pm}\}$. This implies that at least six free parameters are required if the cosmological field equations are to be symmetric under linear transformations of the dynamical variables. Since no more than three are available, however, it follows that no symmetries of the form considered here exist for these two models.

A similar argument applies to the Taub universe [40]. This is a special case of the type IX model and the action is determined by Eqs. (3.6) and (4.2) with $\beta_- = 0$. The kinetic sector of the Taub action is invariant under a continuous $O(2)$ transformation characterized by one free parameter. However, the curvature potential (4.2) contains two independent terms when $\beta_- = 0$, so a minimum of two free parameters are required for the full action to be symmetric. We conclude, therefore, that neither the Bianchi type IX universe nor its isotropic limits (the Taub model and spatially closed FRW cosmology) exhibit duality symmetries that result in non-trivial transformations on the scale factors.

We may also consider the Bianchi type V model (2.8). It follows from Eq. (2.9) that the contribution of the spatial curvature to the type V Lagrangian is of the form ${}^{(3)}Re^{3\alpha-\Phi} \propto e^{\alpha-\Phi}$. Comparison with Eq. (3.2) then implies that the curvature term and the shifted dilaton field cannot simultaneously remain invariant under non-trivial linear transformations of α and Φ . This implies that there is no scale factor duality for this type V model nor its isotropic limit (the $k = -1$ FRW universe). These examples highlight the inhibitive effect of curvature on the existence of dualities.

4.2 Kantowski–Sachs and LRS Bianchi Type III

Rewriting the action (2.12) in terms of the shifted variables (3.1) and (3.2) implies that

$$S = \int dt e^{-\kappa\sigma} \left[\dot{\chi}^2 + \dot{\beta}_1^2 - \dot{\sigma}^2 + 2k \exp \left(C_1\sigma - \frac{2}{\sqrt{6}}(C_4\chi + \beta_1) \right) - 2\Lambda \right], \quad (4.3)$$

where in this subsection $\beta_1 \equiv \sqrt{6}\beta$ and $C_4 \equiv [2/(4 + 3\omega)]^{1/2}$ is a constant. Defining new variables

$$u \equiv C_4\chi + \beta_1 \quad (4.4)$$

$$v \equiv \left(\frac{4 + 3\omega}{6 + 3\omega} \right)^{1/2} (\chi - C_4\beta_1) \quad (4.5)$$

implies that action (4.3) takes the form

$$S = \int dt e^{-\kappa\sigma} \left[\left(\frac{4 + 3\omega}{6 + 3\omega} \right) \dot{u}^2 + \dot{v}^2 - \dot{\sigma}^2 + 2ke^{C_1\sigma - 2u/\sqrt{6}} - 2\Lambda \right]. \quad (4.6)$$

This action is cyclic in v and there exists a discrete Z_2 duality:

$$\bar{u} = u, \quad \bar{v} = -v, \quad \bar{\sigma} = \sigma. \quad (4.7)$$

Eqs. (3.4), (3.5), (4.4) and (4.5) then imply that this is equivalent to

$$\bar{\alpha} = \frac{4 + 3\omega}{3(2 + \omega)}\alpha - \frac{2(1 + \omega)}{3(2 + \omega)}\Phi + \frac{4}{3(2 + \omega)}\beta \quad (4.8)$$

$$\bar{\Phi} = -\frac{2}{2 + \omega}\alpha - \frac{\omega}{2 + \omega}\Phi + \frac{4}{2 + \omega}\beta \quad (4.9)$$

$$\bar{\beta} = \frac{2}{3(2 + \omega)}\alpha + \frac{2(1 + \omega)}{3(2 + \omega)}\Phi + \frac{2 + 3\omega}{3(2 + \omega)}\beta \quad (4.10)$$

in $\{\alpha, \beta, \Phi\}$ space and Eq. (2.11) then implies that the corresponding scale factors transform to

$$\bar{a}_1 = \exp \left[\frac{\omega\alpha - 2(1 + \omega)\Phi - 2\omega\beta}{2 + \omega} \right] \quad (4.11)$$

$$\bar{a}_2 = \exp[\alpha + \beta]. \quad (4.12)$$

As with the Bianchi types VI_0 and VII_0 , one of the scale factors is invariant under the duality transformation because the curvature potential ${}^{(3)}R \propto a_2^{-2}$. Furthermore, when $\omega = -1$, Eqs. (2.11) and (4.11) imply that a_1 is inverted, $\bar{a}_1 = a_1^{-1}$.

5 Discussion and Conclusions

In this paper, we have found a number of discrete and continuous symmetries of the spatially homogeneous, vacuum Brans–Dicke cosmologies containing a cosmological constant in the gravitational sector of the theory. These symmetries relate inequivalent cosmological solutions via linear transformations on the configuration space variables $\{\alpha, \Phi, \beta_+, \beta_-\}$. The dilaton field, Φ , plays a crucial role in the analysis because it transforms in such a way that the shifted field σ may remain invariant even though the cosmological scale factors transform non-trivially.

There exists a discrete Z_2 scale factor duality in the isotropic $k = 0$ FRW model and this symmetry is broken when $k \neq 0$. The duality is extended to a continuous $O(3)$ symmetry for the spatially flat, anisotropic Bianchi type I model when $\omega \neq -4/3$. The symmetry is restricted to $O(2)$ for the type II if $\omega > -11/8$. There is a discrete $Z_2 \times Z_2$ symmetry in the types VI_0 and VII_0 . There are no symmetries of the action that lead to non-trivial transformations of the scale factors in the Bianchi types V, VIII and IX models or the Taub universe. The Kantowski–Sachs and LRS Bianchi type III models exhibit a Z_2 duality. These conclusions are summarized in Table 1. They indicate the extent to which spatial curvature and anisotropy can have inhibitive effects on the allowed forms of duality and this could be of potential significance for the pre-big bang scenario, given that scale factor duality plays a central role in this model.

The action of the continuous $O(3)$ symmetry of the Bianchi type I model corresponds to rotations on the two-sphere $\chi^2 + \beta_1^2 + \beta_2^2 = r^2$, where r is a constant. The spatial curvature of the Bianchi type II model implies that only the subgroup of rotations that leave the variable

	Negative	Flat	Positive
Anisotropic	V	VII₀	IX / T
Isotropic	F ₋₁	F₀	F ₊₁
Anisotropic	III	I	KS

Table 1: A classification of spatially homogeneous Brans–Dicke cosmologies that exhibit duality symmetries, where F_k , KS and T denote the FRW, Kantowski–Sachs and Taub universes, respectively. The boldfaced models are symmetric at some level. The scale factor duality of the flat FRW model can be extended to the anisotropic generalizations of this universe. However, neither the spatially curved $F_{\pm 1}$ models nor their direct anisotropic counterparts are invariant under duality symmetries.

(3.19) invariant represents a symmetry of the Lagrangian. This is formally equivalent to restricting the rotations to the one–surface where the plane $x = C_2\chi + \beta_1 = \text{constant}$ intersects the two–sphere. The intersection corresponds to a circle and the symmetry is therefore reduced to $O(2)$. In the Bianchi types VI_0 and VII_0 , both $|\beta_2|$ and the variable $x = C_3\chi - \beta_1$ must remain invariant. For example, when $\bar{\beta}_2 = \beta_2$, the action of the full group of $O(3)$ rotations must be simultaneously restricted to the two–points on the two–sphere that are intersected by both the $\beta_2 = \text{constant}$ and $x = \text{constant}$ planes. Thus, the symmetry becomes discrete.

An analogous interpretation applies for the Kantowski–Sachs and LRS Bianchi type III models. In these cases, the configuration space is three–dimensional and spanned by $\{\alpha, \Phi, \beta\}$. The kinetic sector of the action is $O(2)$ invariant under arbitrary rotations on the circle $\tilde{u}^2 + v^2 = r^2$, where $\tilde{u} \equiv \sqrt{(4 + 3\omega)/(6 + 3\omega)}u$ and the variable u is defined in Eq. (4.4). When spatial curvature is introduced, the symmetry corresponds to the subgroup of $O(2)$ transformations that also leave the variable $u = C_4\chi + \beta_1$ invariant. As a result, the symmetry of the full action is discrete and represents a map between the two points on the

circle that are intersected by the line $\tilde{u} = \text{constant}$.

The symmetries discussed in this paper have a number of applications. The symmetric nature of the Bianchi class A scalar–tensor cosmologies was recently studied in a different context in Ref. [41]. In that paper the existence of point symmetries in the cosmological field equations was considered. A point symmetry of a set of coupled differential equations may be identified by defining a vector field [38, 42, 43]

$$\mathbf{X} \equiv X_n \frac{\partial}{\partial q_n} + \frac{dX_n}{dt} \frac{\partial}{\partial \dot{q}_n}, \quad (5.1)$$

where $X_n(q)$ are a set of differentiable functions of the configuration space variables q_n . This field belongs to the space that is tangent to the configuration space. Contracting the Euler–Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} = \frac{\partial L}{\partial q_n} \quad (5.2)$$

with the functions $X_n(q)$ then implies that

$$\frac{d}{dt} \left(X_n \frac{\partial L}{\partial \dot{q}_n} \right) = \left(X_n \frac{\partial}{\partial q_n} + \frac{dX_n}{dt} \frac{\partial}{\partial \dot{q}_n} \right) L \quad (5.3)$$

and the Lie derivative of the Lagrangian L with respect to the vector field \mathbf{X} is given by the right hand side of Eq. (5.3). It follows, therefore, that there exists a Noether symmetry when this Lie derivative vanishes and the corresponding conserved quantity is given by

$$\mathcal{C} \equiv X_n \frac{\partial L}{\partial \dot{q}_n}. \quad (5.4)$$

When a cosmological constant is present, it can be shown that point symmetries only exist in the field equations of the Brans–Dicke theory and, moreover, that such symmetries arise in types I, II, VI₀ and VII₀, but *not* for the types VIII and IX [41]. It is interesting that the same class A types admit duality symmetries and this suggests that these dualities may be related to the point symmetries. Indeed, the discrete scale factor duality (3.7) of the spatially flat FRW model may be embedded within a continuous point symmetry, in the

sense that the latter may be employed to derive a new set of configuration space variables that leave the form of the reduced action invariant [41, 44]. A natural question is whether the dualities we have uncovered for the more general Bianchi types are related to point symmetries in a similar way.

It is interesting to consider the behaviour of the first integral (5.4) under duality transformations. In the Bianchi type VII₀, for example, there exists a point symmetry when $X_\Phi = 3X_\alpha = 6X_{\beta_+} = \text{constant}$ and $X_{\beta_-} = 0$ [41] and it follows that

$$\mathcal{C} = 12X_{\beta_+} e^{3\alpha - \Phi} \left[\dot{\alpha} + (1 + \omega)\dot{\Phi} + \dot{\beta}_+ \right]. \quad (5.5)$$

Substitution of Eqs. (3.38)–(3.40) into Eq. (5.5) then implies that this quantity changes sign, $\bar{\mathcal{C}} = -\mathcal{C}$, under the discrete duality transformation. It is therefore invariant if the duality transformation is performed simultaneously with a time reversal, $\bar{t} = -t$.

The duality symmetries we have derived are powerful tools for generating new, inequivalent cosmological solutions from known solutions. For example, when the cosmological constant vanishes, any solution to the vacuum Einstein field equations is also a consistent solution to the Brans–Dicke field equations, where the dilaton is constant. Thus, one may begin with a known homogeneous, Ricci–flat space–time [45] and apply the relevant duality transformations to generate a cosmology with a dynamical dilaton field. The scale factors will also transform non–trivially. These exact ‘dilaton–vacuum’ solutions will allow analytical studies of the pre–big bang scenario to be made in more general anisotropic settings than those considered previously. This will provide valuable insight into the region of parameter space that results in superinflation.

These scale factor dualities of the Bianchi universes also play a central role in studying the effects of a massless scalar axion field, σ , in the matter sector of the Brans–Dicke theory. It proves convenient to perform the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2 \equiv e^{-\phi} \quad (5.6)$$

to the ‘Einstein’ frame, where the dilaton field is minimally coupled. Action (2.1) then takes the form

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{1}{2} \left(\tilde{\nabla} \phi \right)^2 - \frac{1}{2} e^{2\lambda\phi} \left(\tilde{\nabla} \sigma \right)^2 \right] \quad (5.7)$$

where $\phi \equiv (3 + 2\omega)^{1/2}$ and $\lambda \equiv [2(3 + 2\omega)]^{-1/2}$ and we assume that $\Lambda = 0$. Action (5.7) is invariant under a global $\text{SL}(2, \mathbb{R})$ transformation that acts non-linearly on the complex scalar field $\chi \equiv \lambda\sigma + ie^{-\lambda\phi}$ such that the transformed field is given by $\bar{\chi} = (a\chi + b)/(c\chi + d)$, where (a, b, c, d) are constants satisfying $ad - bc = 1$. The Einstein frame metric (5.6) remains invariant under the transformation¹. This implies that the fields ϕ and σ transform non-linearly to

$$e^{\lambda\bar{\phi}} = c^2 e^{-\lambda\phi} + (d + c\lambda\sigma)^2 e^{\lambda\phi} \quad (5.8)$$

$$\lambda\bar{\sigma} e^{\lambda\bar{\phi}} = ace^{-\lambda\phi} + e^{\lambda\phi}(b + a\lambda\sigma)(d + c\lambda\sigma) \quad (5.9)$$

and since the dilaton transforms non-trivially, the Brans–Dicke metric, $g_{\mu\nu}$, transforms to

$$\bar{g}_{\mu\nu} = e^{\bar{\Phi} - \Phi} g_{\mu\nu} \quad (5.10)$$

The importance of this $\text{SL}(2, \mathbb{R})$ symmetry is that cosmological solutions of the form (5.10) with non-trivial σ may be derived from solutions where σ is constant, i.e., from the dilaton–vacuum solutions discussed above. Thus, the scale factor dualities of the Bianchi models provide the first step in generating anisotropic cosmologies with non-trivial dilaton and matter sector from vacuum solutions to general relativity. The role of massless scalar fields on the scenario may therefore be studied analytically and possible observational effects of quantum fluctuations in the matter fields may also be discussed along the lines developed in Refs. [46, 47].

Duality symmetries also have important applications in quantum cosmology. There exists a well known factor ordering problem in the standard approach due to ambiguities

¹The reader is referred to Refs. [46, 47] for details.

in the quantization of the classical variables. In the spatially flat FRW Brans–Dicke model, however, the scale factor duality (3.7) may be employed to resolve this problem [7]. The natural ordering to choose is the one where the Wheeler–DeWitt equation remains invariant under the duality transformation. In principle, the duality symmetries of the anisotropic cosmologies could be employed in a similar way when quantizing these models. We also remark that the scale factor duality of the flat FRW universe is related to a hidden $N = 2$ supersymmetry at the quantum level [34]. This leads to a supersymmetric approach to quantum cosmology that may resolve the problems encountered in defining a non-negative norm for the wavefunction of the universe [48, 49]. It would be of interest to investigate whether the dualities of the Bianchi models are related to hidden supersymmetries.

Finally, it would be interesting to generalize the analysis to non-cosmological spacetimes. Cadoni has shown that $(2 + 1)$ -dimensional Brans–Dicke gravity is $O(2)$ invariant when the metric is static and circularly symmetric [35]. The discrete $O(2, \mathbb{Z})$ subgroup of $O(2)$ was employed to generate a new spacetime with a conical singularity that is dual to the black string solution of Ref. [50]. The question arises as to whether an $O(3)$ symmetry exists in the four-dimensional Brans–Dicke theory when the metric is static.

Acknowledgments

DC and JEL are supported by the Particle Physics and Astronomy Research Council (PPARC), UK. RT benefited from SERC UK Grant No. H09454. We thank JWB Hughes and MAH MacCallum for helpful discussions.

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